

NON-LEFT-COMPLETE DERIVED CATEGORIES

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ABSTRACT. We give some examples of abelian categories \mathcal{A} for which the derived category $\mathbf{D}(\mathcal{A})$ is not left-complete. Perhaps the most natural of these is where \mathcal{A} is the category of representations of the additive group \mathbb{G}_a over a field k of characteristic $p > 0$.

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0. ASSUMED BACKGROUND

In this article we assume the reader is familiar with derived categories and with t -structures on them. See Verdier [5] for the theory of derived categories, and Beilinson, Bernstein and Deligne [1, Chapter 1] for an introduction to t -structures.

1. THE COUNTEREXAMPLE

Suppose \mathcal{A} is an abelian category and $\mathbf{D}(\mathcal{A})$ is its derived category. For any object $x \in \mathbf{D}(\mathcal{A})$, we write $x^{\geq n}$ for the truncation of x with respect to the standard t -structure. We have canonical maps $x^{\geq n} \rightarrow x^{\geq n+1}$, and a (non-canonical) map

$$\varphi_x : x \longrightarrow \varinjlim x^{\geq n}.$$

The category $\mathbf{D}(\mathcal{A})$ is said to be *left-complete* if, for every object $x \in \mathbf{D}(\mathcal{A})$, any map φ_x as above is an isomorphism. Even though the map φ_x is not canonical, it can be shown that, for given x , if one φ_x is an isomorphism then they all are.

The reader can find much more about left-complete categories in Lurie [3, Section 7] or [4, Subsection 1.2.1, more precisely starting from Proposition 1.2.1.17]. See also Drinfeld and Gaitsgory [2].

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In this note we will see how to produce many \mathcal{A} for which $\mathbf{D}(\mathcal{A})$ is not left-complete. Our counterexamples will be of a very special form, which allows us to easily compute the homotopy inverse limit $\varprojlim x^{\geq n}$. Let us now sketch what we will do.

We will suppose that the abelian category \mathcal{A} satisfies the axiom [AB4], that is coproducts are exact; this makes it easy to compute coproducts in the derived category $\mathbf{D}(\mathcal{A})$, just form the coproduct as complexes. Suppose A is an object in our [AB4] abelian category \mathcal{A} , and let

$$x = \prod_{i=0}^{\infty} A[i] .$$

It is clear that, for $n > 0$, we have

$$x^{\geq -n} = \prod_{i=0}^n A[i] = \prod_{i=0}^n A[i] ,$$

where the last equality is because finite coproducts agree with finite products. Now the homotopy inverse limit of the products is a genuine inverse limit, and we have

$$\varprojlim x^{\geq n} = \prod_{i=0}^{\infty} A[i] .$$

Thus our problem becomes to decide whether the map

$$\prod_{i=0}^{\infty} A[i] \xrightarrow{\varphi} \prod_{i=0}^{\infty} A[i]$$

is an isomorphism. Note that in this case the map is canonical; our homotopy inverse limit happens to be a genuine inverse limit, removing the arbitrariness. The left hand side is easy to work with; its cohomology is A in each degree $n \leq 0$. What we will show is how to produce examples where the right hand side has lots more cohomology. More precisely, we have

$$\prod_{i=0}^{\infty} A[i] = A[0] \oplus \left(\prod_{i=1}^{\infty} A[i] \right)$$

and the expectation would be for the second term to have a vanishing H^0 ; what we will show is how to produce non-zero classes in

$$H^0 \left(\prod_{i=1}^{\infty} A[i] \right) .$$

It is time to disclose what will be our choice for the category \mathcal{A} and for the object $A \in \mathcal{A}$.

Construction 1.1. Let k be a field, let R_1 be a finitely generated k algebra, and let \mathfrak{m} be a k -point of $\mathrm{Spec}(R_1)$. In other words, $\mathfrak{m} \subset R$ is a maximal ideal with $R_1/\mathfrak{m} \cong k$. We make a string of definitions:

- (i) $R_n = \otimes_{i=1}^n R_1$, where the tensor is over the field k .

- (ii) The inclusion $R_n \longrightarrow R_{n+1}$ is the inclusion of the tensor product of the first n terms.
- (iii) $R = \varinjlim R_n$.
- (iv) The map $\Phi_i : R_1 \longrightarrow R$ is the inclusion of the i^{th} factor.
- (v) The category \mathcal{A} will be the category of all those R -modules, on which $\Phi_i(\mathfrak{m})$ acts trivially for all but finitely many i .

The object $A \in \mathcal{A}$ will be the colimit over n of the R_n -modules $k = \otimes_{i=1}^n [R_1/\mathfrak{m}]$.

The main result is

Theorem 1.2. *Assume that $k = R_1/\mathfrak{m}$ is not projective over the localization $(R_1)_{\mathfrak{m}}$ of the ring R_1 at the maximal ideal \mathfrak{m} . With the category \mathcal{A} and the object $A \in \mathcal{A}$ as in Construction 1.1, there is a non-zero element in*

$$H^0 \left(\prod_{i=1}^{\infty} A \right) .$$

Remark 1.3. The case where $R_1 = k[x]/(x^p)$ is of particular interest. If the field k is of characteristic p then the category \mathcal{A} happens to be the category of representations of the additive group \mathbb{G}_a , and we learn that its derived category is not left-complete.

Remark 1.4. We trivially have

$$\prod_{i=1}^{\infty} A[i] = \left(\prod_{i=1}^n A[i] \right) \oplus \left(\prod_{i=n+1}^{\infty} A[i] \right) ,$$

and hence

$$H^0 \left(\prod_{i=1}^{\infty} A[i] \right) = H^0 \left(\prod_{i=1}^n A[i] \right) \oplus H^0 \left(\prod_{i=n+1}^{\infty} A[i] \right) .$$

On the other hand, with the finite product we have no problem computing

$$H^0 \left(\prod_{i=1}^n A[i] \right) = H^0 \left(\prod_{i=1}^n A[i] \right) = 0 ,$$

and Theorem 1.2 now allows us to deduce that

$$H^0 \left(\prod_{i=n+1}^{\infty} A[i] \right) \neq 0 .$$

Translating we have that

$$H^n \left(\prod_{i=1}^{\infty} A[i] \right) \neq 0$$

for all $n \geq 0$. The complexes $A[i]$, $i > 0$ all belong to $\mathbf{D}(\mathcal{A})^{<0}$, but the product $\prod_{i=1}^{\infty} A[i]$ is not bounded above.

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2. THE PROOF

We begin with a little lemma.

Lemma 2.1. *Let k be a field, and let R and S be finitely generated k -algebras. Suppose further that we are given k -points of $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$; that is $\mathfrak{m} \subset R$ and $\mathfrak{n} \subset S$ are maximal ideals, with*

$$R/\mathfrak{m} \cong k \cong S/\mathfrak{n}.$$

Let E be an injective envelope of $k = R/\mathfrak{m}$ over the ring R , and F an injective envelope of $k = S/\mathfrak{n}$ over the ring S . Then $E \otimes_k F$ is an injective envelope of k over the ring $R \otimes_k S$.

Proof. We will first prove the case where R and S are polynomial rings.

Let $R' = k[x_1, x_2, \dots, x_m]$ be a polynomial ring, and let \mathfrak{m} be the maximal ideal generated by $\{x_1, x_2, \dots, x_m\}$. Then we know the injective envelope E' of $k = R'/\mathfrak{m}$ explicitly: it is the quotient of $S = k[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_m, x_m^{-1}]$ by the R' -submodule generated by all monomials $x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$ with at least one of the $i_j > 0$. As a k -vector space $E' = k[x_1^{-1}, x_2^{-1}, \dots, x_m^{-1}]$, and the R' -module structure is obvious when we declare $x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} = 0$ if some $i_j > 0$. If $S' = k[y_1, y_2, \dots, y_n]$ and $\mathfrak{n} \subset S'$ is the ideal generated by $\{y_1, y_2, \dots, y_n\}$, then the fact that

$$E' \otimes_k F' = k[x_1^{-1}, x_2^{-1}, \dots, x_m^{-1}] \otimes_k k[y_1^{-1}, y_2^{-1}, \dots, y_n^{-1}]$$

is the injective hull of k over $R' \otimes S'$ is by inspection.

Now for the general case: assume $R = R'/I$ and $S = S'/J$ where R' and S' are polynomial rings, and $I \subset R'$ and $J \subset S'$ are ideals contained in the \mathfrak{m} and \mathfrak{n} above. Then the injective hull E of $k = R/\mathfrak{m}$ over the ring R is the largest R -submodule of the R' -module E' , that is the R' -submodule $E \subset E'$ of all elements annihilated by the ideal I . The lemma therefore comes down to the fact that the submodule of $E' \otimes_k F'$ annihilated by the ideal $I \otimes_k S' + R' \otimes_k J$ is precisely $E \otimes_k F$. \square

Proof of Theorem 1.2. Let \overline{R} be the localization of R_1 at the maximal ideal \mathfrak{m} . We are assuming that k is not projective over \overline{R} , that is the projective dimension of k is at least one. Choose and fix a minimal free resolution of $k = \overline{R}/\mathfrak{m}\overline{R}$ as an \overline{R} -module. Let us write this resolution as

$$\longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0.$$

Then the modules P_i are all finite and free over the ring \overline{R} , the differentials are all matrices over \overline{R} , and the minimality guarantees that the entries in these matrices all belong to the ideal $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R} \subset \overline{R}$. Now let E be the \overline{R} -injective envelope of the module k ; applying the functor $\mathrm{Hom}_{\overline{R}}(-, E)$ to the projective resolution above, we produce an injective resolution I^* of k , which we write out as

$$0 \longrightarrow k \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow$$

We know that each $I^j = \text{Hom}(P_j, E)$ is a finite coproduct of copies of E , and that the differentials $I^j \rightarrow I^{j+1}$ are matrices whose entries belong to the ideal $\overline{\mathfrak{m}}$. The fact that the projective dimension of k is at least one tells us that $P_1 \neq 0$, and therefore $I^1 \neq 0$. Note that an injective envelope E of k over the localized ring $\overline{R} = (R_1)_{\mathfrak{m}}$ is also an injective envelope of k over the ring R_1 , hence we have produced an injective resolution of k over R_1 . Next we

- (i) Choose a non-zero element a in the image of the map $k \rightarrow I^0$.
- (ii) Choose a non-zero element $b \in I^1$, with $\mathfrak{m}b = 0$.

If we view k as a module over the ring $R_n = \otimes_{i=1}^n R_1$, then the tensor product $J_n^* = \otimes_{i=1}^n I^*$ is certainly a resolution of k as an R_n module, and Lemma 2.1 guarantees further that

- (iii) Each J_n^i is injective as a module over R_n .
- (iv) Let the inclusion $J_n^* \rightarrow J_{n+1}^*$ be the map taking $x \in J_n^*$ to

$$x \otimes a \in J_n^* \otimes I^0 \subset J_n^* \otimes I^* = J_{n+1}^*,$$

where $a \in I^0$ is as in (i) above. We define J^* to be

$$J^* = \varinjlim J_n^*;$$

then J^* is an injective resolution of k in the category \mathcal{A} .

To prove the theorem we need to find a non-zero element in $H^0(\prod_{i>0} k[i])$, and our next observation is that the product in the derived category $\prod_{i>0} k[i]$ is obtained as the ordinary product of injective resolutions. The complex $J^*[i]$ is an injective resolution of $k[i]$, and hence the derived product $\prod_{i>0} k[i]$ is just the usual product $\prod_{i>0} J^*[i]$. Now for every $i \geq 1$ let

$$S_i = \{i^2 + 1, \dots, i^2 + i\},$$

and observe that the sets S_i are disjoint. In the injective R_{i^2+i} -module

$$J_{i^2+i}^i = \coprod_{\sum \ell_m = i} I^{\ell_1} \otimes I^{\ell_2} \otimes \dots \otimes I^{\ell_{i^2+i}}$$

or more specifically in the summand

$$(I^0)^{\otimes i^2} \otimes (I^1)^{\otimes i}$$

we take the term

$$\lambda_i = a^{\otimes i^2} \otimes b^{\otimes i},$$

where $a \in I^0$ and $b \in I^1$ are as in (i) and (ii) above. The embedding $J_{i^2+i}^* \rightarrow J^*$ of (iv) gives us an element which we will denote $\lambda_i \in J^i$. The elements λ_i have the properties

- (v) Each λ_i is a cycle; the differential $J^i \rightarrow J^{i+1}$ kills λ_i .
- (vi) $\Phi_j(\mathfrak{m})\lambda_i = 0$ for all i and j .

We are assuming $i > 0$, so each λ_i must be a boundary because $H^i(J^*) = 0$. But if $\mu_i \in J^{i-1}$ maps to λ_i , then there must exist an integer $j \in S_i$ so that $\Phi_j(\mathfrak{m})$ does not kill μ_i . Now form the element

$$\prod_{i=1}^{\infty} \lambda_i \in \prod_{i=1}^{\infty} J^i,$$

where the product is in the category of all R -modules.

Caution 2.2. The reader is reminded that the category \mathcal{A} is a subcategory of the category of R -modules. Both categories have infinite products; the products in the category of R -modules are just the usual cartesian products, while the products in \mathcal{A} are subtler. To form the product in \mathcal{A} of a bunch of objects in \mathcal{A} , one first forms the usual cartesian product, and then consider inside it the largest object belonging to \mathcal{A} , that is the collection of all elements satisfying part (v) of Construction 1.1.

The element $\prod_{i=1}^{\infty} \lambda_i$ is a degree 0 cycle in the complex $\prod_{i \geq 1} J^*[i]$, and it is annihilated by $\Phi_j(\mathfrak{m})$ for all j . By Caution 2.2 we have that $\prod_{i=1}^{\infty} \lambda_i$ belongs to $\prod_{i=1}^{\infty} J^i$ even when the product is understood in \mathcal{A} . However, it is not a boundary in \mathcal{A} . If we try to express $\prod_{i=1}^{\infty} \lambda_i$ as the boundary of

$$\prod_{i=1}^{\infty} \mu_i \in \prod_{i=1}^{\infty} J^{i-1},$$

then we discover that each μ_i fails to be annihilated by some $\Phi_j(\mathfrak{m})$ with $j \in S_i$. As the S_i are disjoint, this produces infinitely many $\Phi_j(\mathfrak{m})$ not annihilating $\prod_{i=1}^{\infty} \mu_i$, meaning it does not belong to \mathcal{A} .

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